COMPLETE ALGEBRAIC VECTOR FIELDS ON DANIELEWSKI SURFACES

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ABSTRACT. We give the classification of all complete algebraic vector fields on Danielewski surfaces (smooth surfaces given by $xy = p(z)$). We use the fact that for each such vector field there exists a certain fibration that is preserved under its flow. In order to get the explicit list of vector fields a classification of regular function with general fiber $\mathbb{C}$ or $\mathbb{C}^*$ is required. In this text we present results about such fibrations on Gizatullin surfaces and we give a precise description of these fibrations for Danielewski surfaces.

1. INTRODUCTION

Complete (= globally integrable) vector fields are vector fields for which a global holomorphic flow map exists. In general the problem of classifying complete vector fields on Stein manifolds seems to be out of reach. However, for complete algebraic vector fields on affine varieties there are some known results. In 2000 Andersén [1] gave a classification of complete algebraic vector fields on $(\mathbb{C}^*)^n$. For affine surfaces the situation looks better. In 2004 Brunella [6] gave an explicit classification of complete algebraic vector fields on $\mathbb{C}^2$. The proof uses deep results from the theory of foliations on projective surfaces developed by Brunella [4,5], McQuillan [14], and others. From this theory it follows that there is always a regular function with general fibers isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$ such that the vector field sends fibers to fibers. Since these functions on $\mathbb{C}^2$ where classified by Suzuki [15] it was only a small step to conclude the explicit form of the complete algebraic vector fields on $\mathbb{C}^2$. An extension of this result to affine toric surfaces (a quotient of $\mathbb{C}^2$ by some cyclic group action) has been recently presented in [12]. The fact that each complete algebraic vector field preserves the fibers of a regular function with $\mathbb{C}$ or $\mathbb{C}^*$ fibers turns out to be true on almost all normal affine surfaces. This makes it possible to classify all complete algebraic vector fields for other surfaces.

Fact ([11, Theorem 1.3])**.** *Let S be a normal affine surface such that not all complete algebraic vector fields on S are proportional, and let ν be a complete algebraic vector field on S. Then there exists a regular function $f : S \to \mathbb{C}$ with general fiber isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$ such that the flow of ν sends fibers of f to fibers of f (in short: ν preserves the fibration f).*

This fact shows that once the classification of $\mathbb{C}$- and $\mathbb{C}^*$-fibrations is done the complete vector fields are described. In this text we give some results about $\mathbb{C}$- and $\mathbb{C}^*$-fibrations on Gizatullin surfaces. For the special case of smooth surfaces

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given by $xy = p(z)$ (which are called Danielewski surfaces) we can provide a precise classification: Here we give the list of complete algebraic vector fields. Since the $\mathbb{C}$- and $\mathbb{C}^*$-polynomials on Danielewski surfaces look much alike the ones on $\mathbb{C}^2$ the vector fields also look similarly. Surprisingly if $\deg(p) = 4$ there occurs a complete vector field that has no analogue on $\mathbb{C}^2$.

Section 2 is a recapitulation of the definition of Gizatullin surfaces, a generalization of Danielewski surfaces, and SNC-completions, a powerful tool for affine algebraic surfaces. In section 3 we present some results about $\mathbb{C}$- and $\mathbb{C}^*$-fibrations on Gizatullin surfaces which will be used in section 4 to give an explicit description of $\mathbb{C}$- and $\mathbb{C}^*$-fibrations on Danielewski surfaces. Section 5 combines this description to a proof of the following theorem:

Main Theorem. *Let ν be a complete algebraic vector field on $S = \{xy = p(z)\}$ (where p has simple zeros), and let the hyperbolic vector field (HF) and the two shear vector fields (SF) be defined as follows:*

$$\mathrm{HF} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \mathrm{SF}^x = p'(z)\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \quad \text{and} \quad \mathrm{SF}^y = p'(z)\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

Then ν occurs in the following list (up to an automorphism of S):

(1) ν preserves the polynomial x and is of the form:

$$\nu = c\mathrm{HF} + (A(x)z + B(x))\,\mathrm{SF}^x$$

for some $c \in \mathbb{C}$ and $A, B \in \mathbb{C}[x]$.

(2) ν preserves a polynomial $x^m(x^l(z+a)+Q(x))^n$ for coprime numbers $m, n \in \mathbb{N}$, $\deg(Q) \in \mathbb{N}_0$, $a \in \mathbb{C}$ and $\deg(Q) < l$ and is of the form:

$$\begin{aligned}\nu \quad &= \quad c\left(\frac{z+a}{x} + \frac{Q(x)}{x^{l+1}}\right)\mathrm{SF}^x + A(x^m(x^l(z+a)+Q(x))^n) \\ &\quad \cdot \left[n\mathrm{HF} - \left(\frac{(m+nl)(z+a)}{x} + \frac{mQ(x)+nxQ'(x)}{x^{l+1}}\right)\mathrm{SF}^x\right]\end{aligned}$$

for some $c \in \mathbb{C}$ and $A \in \mathbb{C}[t]$ satisfying $A(0) = c/(m+nl)$ and $A(x^m(x^l(z+a)+Q(x))^n)(mQ(x)+nxQ'(x)) - cQ(x) \in x^{l+1}\cdot\mathbb{C}[S]$.

(3) If $\deg(p) = 4$ then ν can also preserve the polynomial $ax + y + \frac{1}{6}p''(z)$ where a is the leading coefficient of p. In this case ν looks like:

$$\nu = A\left(ax + y + \frac{1}{6}p''(z)\right)\left(-\frac{1}{6}p'''(z)\mathrm{HF} + a\mathrm{SF}^x - \mathrm{SF}^y\right)$$

for some $A \in \mathbb{C}[t]$.

The Main Theorem describes a class of one-parameter subgroups of the group of holomorphic automorphisms on $S = \{xy = p(z)\}$. It is worth to compare this result to well known results in the algebraic case. Daigle [7] and Makar-Limanov [13] showed that on S every algebraic $\mathbb{C}^+$-action is up to an algebraic automorphisms induced by some vector field $f(x)\mathrm{SF}^x$ for some polynomial $f \in \mathbb{C}[x]$, and thus is a special case of (1) of the Main Theorem. Moreover, by [9] there is a unique (up to an automorphism) algebraic $\mathbb{C}^*$-action on S which is induced by HF, which can be seen as a vector field of type (1) or (2).

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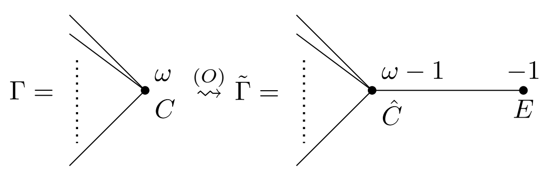
2. Gizatullin surfaces and their completions

2.1. **SNC-completions and dual graphs.** It is a well established procedure in affine algebraic geometry to use so called SNC-completions of affine surfaces. Let S be an affine surface, and let $X \supset S$ be a projective surface such that the *boundary divisor* $D = X \setminus S = C_1 \cup \ldots \cup C_k$ is contained in the smooth locus of X. If moreover the curves C_i are smooth and intersect pairwise transversally and at most in double points of D then we say that X is a completion of S with *simple normal crossings* (in short: *SNC-completion*). Every normal affine surface admits an SNC-completion. In this text $D = X \setminus S$ will always be a union of rational curves.

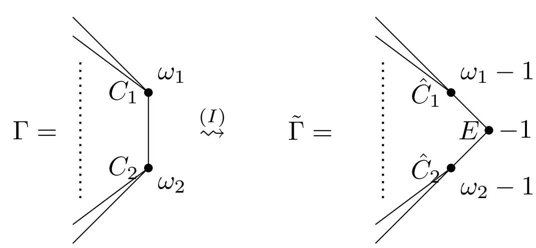
A good reference for SNC-completions is for example [8]. In particular, in this reference most notions that are used in this section are introduced. However the concept of SNC-completions was already used much earlier by Danilov and Gizatullin. Let X be an SNC-completion of an affine surface S then its *dual graph* Γ_X is given as follows: The vertices of Γ_X are given by the irreducible components C_i of the boundary $D = X \setminus S$ and each intersection point $p \in C_i \cap C_j$ of two different components corresponds to an edge of Γ_X that connects the vertices which correspond to C_i and C_j. The graph Γ_X is often considered as a weighted graph where the weight of a vertex is given by the self-intersection $C_i \cdot C_i$ of its corresponding curve C_i.

Clearly neither SNC-completions nor dual graphs are unique: Modifications along the boundary will change the boundary and the dual graph of the boundary. The following two modifications (and its inverses) are possible (C is the name of the vertex and $\omega = C \cdot C$ is its weight):

Outer blow up:
(of a point on a curve C)



Inner blow up:
(of a point on $C_1 \cap C_2$)



where E is the exceptional divisor and $\hat{C}$ denotes the strict transform of a curve C. A sequence of (I), (I^{-1}), (O) and (O^{-1}) starting with a weighted graph is called a modification of weighted graphs. A birational map $\varphi : X \dashrightarrow Y$ between two completions X, Y of an affine surface S such that $\varphi|_S$ induces an isomorphism on S is called a birational modification of completions and an isomorphism of completions if φ is additionally an isomorphism. By a classical theorem of Zariski any birational map can be seen as composition of blow ups followed by a composition of blow downs. Hence we get the following statement:

Theorem of Zariski. *(1) Let S be an affine surface and let X and Y be two SNC-completions. Then there exists a SNC-completion Z of S obtained via a sequence of blow ups performed over the boundaries of S in X and Y, respectively. Hence Γ_Z is obtained by modifications as above from both Γ_X and Γ_Y.*

(2) Let $\gamma : \Gamma_X \rightsquigarrow \Gamma$ be a modification of weighted graphs. Then there is a completion Y of S such that $\Gamma_Y = \Gamma$ and Y is obtained from X by a birational map $\phi : X \dashrightarrow Y$ that induces the modification γ on the dual graphs. If γ does not contain outer blow ups then ϕ is uniquely determined.

A completion X will be called minimal if Γ_X does not have a (-1)-vertex of degree ≤ 2.

2.2. **Gizatullin surfaces.** A *Gizatullin surface* is a normal affine surface S that admits an SNC-completion X such that the graph Γ_X is linear. For such a completion (also called a *zigzag*) with

$$\Gamma_X = \quad \begin{array}{ccccc} \omega_0 & \omega_1 & & & \omega_k \\ \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\!\!\! & \bullet\!\!-\!\!-\!\!-\!\!-\!\!- & \cdots\cdots & -\!\!-\!\!-\!\!-\!\!- & \bullet \\ C_0 & C_1 & & & C_k \end{array}$$

we use the notation

$$\Gamma_X = [[\omega_0, \omega_1, \ldots, \omega_k]].$$

A completion X is called *standard* if

$$\Gamma_X = [[0, 0, \omega_2, \ldots, \omega_k]] \quad \text{or} \quad \Gamma_X = [[0, 0, 0]] \quad \text{or} \quad \Gamma_X = [[0, 0]]$$

and ω_1-*semistandard* if

$$\Gamma_X = [[0, \omega_1, \omega_2, \ldots, \omega_k]] \quad \text{or} \quad \Gamma_X = [[0, \omega_1, 0]] \quad \text{or} \quad \Gamma_X = [[0, \omega_1]]$$

with $\omega_i \leq -2$ for all $2 \leq i \leq k$.

Now we introduce two modifications of the boundary of Gizatullin surfaces. The first one is

$$\text{(A)} \qquad [[0, \omega_1, \ldots]] \overset{(O)}{\rightsquigarrow} [[-1, -1, \omega_1, \ldots]] \overset{(I^{-1})}{\rightsquigarrow} [[0, \omega_1 + 1, \ldots]]$$

that allows to transform any semistandard completion X with $\Gamma_X = [[0, \omega_1, \omega_2, \ldots]]$ into a standard completion Y with $\Gamma_Y = [[0, 0, \omega_2, \ldots]]$. The second modification is a way to use a zero vertex in order to move weight from one side of the vertex to the other:

$$\text{(B)} \qquad [[\ldots, \omega_{i-1}, 0, \omega_{i+1} \ldots]] \overset{(I)}{\rightsquigarrow} [[\ldots [\omega_{i-1}, -1, -1, \omega_{i+1} - 1, \ldots]]$$

$$\overset{(I^{-1})}{\rightsquigarrow} [[\ldots, \omega_{i-1} + 1, 0, \omega_{i+1} - 1, \ldots]]$$

By a sequence of modification of type (B) it is possible to move zeros vertices through the boundary divisor:

$$[[0, 0, \omega_2, \ldots, \omega_k]] \rightsquigarrow [[\omega_2, 0, 0, \ldots, \omega_k]] \rightsquigarrow \cdots \rightsquigarrow [[\omega_2, \ldots, \omega_k, 0, 0]].$$

The modification above is called *reversion* and it shows that the data of a standard completion is in general not unique. Using these modifications we see that each Gizatullin surface admits a standard completion and that all minimal completions have a linear dual graph:

Proposition 2.1 ([8]). *Let S be a Gizatullin surface. Then:*

(1) There exists a standard completion X, and the dual graph Γ_X is unique up to reversion.

(2) For any completions X there is a contraction (i.e. a modification consisting of (O^{-1}) and (I^{-1})) of Γ_X to a linear graph.

3. $\mathbb{C}$- and $\mathbb{C}^*$-fibrations on Gizatullin surfaces

Let us start with some well know facts in algebraic geometry.

Proposition 3.1. *(a) [2] Let S be a normal affine surface, and let $f : S \to \mathbb{C}$ be a reduced*[1] *regular function with rational fibers. Then there is a pseudo-minimal SNC-completion X such that f extends to a regular function $\bar{f} : X \to \mathbb{P}^1$ with general fibers isomorphic to $\mathbb{P}^1$.*

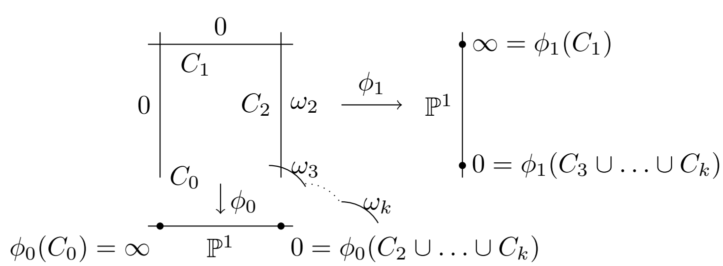
(b) [15] Let f be as in (a) then $\chi(S) = \chi(F \times \mathbb{C}) + \sum(\chi(F') - \chi(F))$ where χ denotes the Euler characteristic, F is a regular fiber of f and the sum is taken over all singular fibers F'.

(c) [2] Let X be a smooth projective surface and let $f : X \to \mathbb{P}^1$ a regular function with general fiber isomorphic to $\mathbb{P}^1$. Then there is a sequence of contractions $\pi : X \to Y$ and a map $f' : Y \to \mathbb{P}^1$ such that $f = f' \circ \pi$ and f' is a $\mathbb{P}^1$-bundle.

(d) [9] Let $C \cong \mathbb{P}^1$ be a curve on a rational projective surface X with $C \cdot C = 0$ then there is a regular function $f : X \to \mathbb{P}^1$ such that $C = f^{-1}(\infty)$ is a regular fiber of f.

Definition 3.2. If a regular function $f : S \to \mathbb{C}$ (or $\mathbb{P}^1$) on a variety S is considered as *fibration* it means that we are only interested in its level sets (i.e. the fibers). In particular, two regular functions are considered to be the same fibrations whenever the differ only by a Möbius transform in the target. A fibration f is said to be a $\mathbb{C}$- (resp. $\mathbb{C}^*$- or $\mathbb{P}^1$-) fibration if its regular fiber is isomorphic to $\mathbb{C}$ (resp. $\mathbb{C}^*$ or $\mathbb{P}^1$).

Let S be a Gizatullin surface, and let X be a standard completion with boundary $D = X \setminus S = C_0 \cup \ldots \cup C_k$. Then the two curves C_0 and C_1 induce (Proposition 3.1(d)) both a regular function $\phi_0, \phi_1 : X \to \mathbb{P}^1$ such that $C_0 = \phi_0^{-1}(\infty)$ and $C_1 = \phi_1^{-1}(\infty)$ are regular fibers. The function ϕ_0 (resp. ϕ_1) is constant on C_i for $2 \le i \le k$ (resp. $3 \le i \le k$), and we may assume that it is vanishing there. Moreover ϕ_0 (resp. ϕ_1) restricted to C_1 (resp. C_0 and C_2) is an isomorphism.



Hence the map $\phi = \phi_0 \times \phi_1 : X \to \mathbb{P}^1_x \times \mathbb{P}^1_y$ induces isomorphisms $\phi|_{C_0} : C_0 \to \{x = \infty\}$, $\phi|_{C_1} : C_1 \to \{y = \infty\}$ and $\phi|_{C_2} \to \{x = 0\}$, and moreover ϕ contracts the curves $C_3, \ldots, C_k$ onto $(0, 0)$. Altogether the map ϕ describes a way how to construct a Gizatullin surface starting with $\mathbb{C}^2$ and blowing up points on $\{x = 0\}$.

[1]Recall that a regular function $f : S \to \mathbb{C}$ is called reduced if its general fiber is connected.

The exceptional divisor of ϕ consists of the curves $C_3, \ldots, C_k$ and additional curves (called feathers) $F_1, \ldots, F_n$ that intersect the surface S. By Proposition 3.1(b) the number of feathers is precisely $\chi(S)$.

Now, we are able to state some results about rational fibrations on Gizatullin surfaces. Propositions 3.3, 3.5 and 3.7 are specializations of Proposition 6.6 in [11]. In order to be self-contained we still present complete proofs. Let us start with $\mathbb{C}$-fibrations.

Proposition 3.3 ([9])**.** *Let $f : S \to \mathbb{C}$ be a $\mathbb{C}$-fibration on a Gizatullin surface S. Then there is a standard completion X such that f coincides with the fibration ϕ_0 given as above.*

Proof. Let X be a pseudo-minimal SNC-completion of S such that f extends to a regular function $\bar{f}$. A general fiber of $\bar{f}$ intersects $D = X \setminus S = C_0 \cup \ldots \cup C_k$ in precisely one point, therefore one curve in D (say C_1) is a section of $\bar{f}$, and on every other curve in D the function $\bar{f}$ is constant. The set $f^{-1}(\infty) \subset D$ is contractible to a rational curve (apply 3.1(c) to a the desingularisation of X) which intersects C_1 transversally (since C_1 is a section). So by pseudo-minimality $\bar{f}^{-1}(\infty)$ is already an irreducible curve (say C_0) with self-intersection 0. Moreover, by the absence of further sections, C_0 is disjoint from $C_2, \ldots, C_k$. Assume that the dual graph Γ_X is not linear and let C_i be a vertex of degree ≥ 3. By Proposition 4.2(2) all but two branches at C_i are contractible, but by pseudo-minimality the only branch that could be not minimal is the one containing C_1. On the other hand, the branch containing C_1 cannot be contractible since it contains also the vertex C_0, which has weight 0, and thus is not contractible. Altogether Γ_X is linear and of the form $\Gamma_X = [[0, n, \omega_2, \ldots, \omega_k]]$ with n arbitrary and $\omega_i \leq -2$, and can be transformed using the modification (A) into a standard completion such that the fibration ϕ_0 coincides with $\bar{f}$. $\square$

Corollary 3.4 ([9])**.** *For a Gizatullin surface S there are as many $\mathbb{C}$-fibrations up to an automorphism as there are standard completions of S up to an isomorphism.*

Note that there are families of Gizatullin surfaces that have a unique standard completion up to reversion. Check [9] for a description of such Gizatullin surfaces. The surfaces called Danielewski surfaces that are introduced in Section 4 are of this kind. Later, Corollary 3.4 will be used e.g. in Proposition 4.2.

For $\mathbb{C}^*$-fibrations there are two different cases: The fibration could have either two sections at infinity or one double-section at infinity (i.e. a curve such that the fibration restricted to this curve is a ramified 2-sheeted covering). First we deal with the case when there are two sections.

Proposition 3.5. *Let $f : S \to \mathbb{C}$ be a $\mathbb{C}^*$-fibration on a Gizatullin surface S, and let Y be a pseudo-minimal SNC-completion of S such that the boundary divisor $Y \setminus S$ contains two sections.*

(1) We may choose Y such that the dual graph is of the form

$$\Gamma_Y = \begin{array}{ccccccccccccc} \eta_{-m} & & \eta_{-2} & & 0 & & 0 & & \eta_1 & & \eta_2 & & \eta_n \\ \bullet & \cdots & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & \cdots & \bullet \\ D_{-m} & & D_{-2} & & D_{-1} & & D_0 & & D_1 & & D_2 & & D_n \end{array}$$

with $m, n \geq 0$, $\eta_1 \leq -1$ and $\eta_i \leq -2$ for $|i| \geq 2$ and additionally $D_0 = \bar{f}^{-1}(\infty)$.

(2) There is a ω_1-semistandard completion $X \supset S$ with $\omega_1 \geq 0$ and

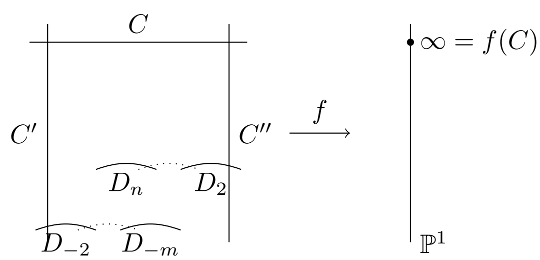
$$\Gamma_X = \quad \begin{array}{ccccccc} 0 & & \omega_1 & & \omega_2 & & \omega_k \\ \bullet & \text{---} & \bullet & \text{---} & \bullet & \cdots & \bullet \\ C_0 & & C_1 & & C_2 & & C_k \end{array}$$

such that Y is obtained from X by (i) a sequence of inner (unless $S = \mathbb{C}^2$) blow ups at infinitely near points followed by (ii) a modification of type (B):

$$[[0, \omega_1, \omega_2, \ldots, \omega_k]] \overset{(i)}{\rightsquigarrow} [[0, 0, \eta_{-m}, \ldots, \eta_{-2}, \eta_1, \eta_2, \ldots, \eta_n]]$$

$$\overset{(ii)}{\rightsquigarrow} [[\eta_{-m}, \ldots, \eta_{-2}, 0, 0, \eta_1, \ldots, \eta_n]].$$

Proof. The proof of (1) works very similarly to the proof above. Again, by Proposition 3.1(c), $\bar{f}^{-1}(\infty)$ is contractible so a curve C with self-intersection equal to 0 and the two sections C' and C'' intersect C transverally since they are sections. Moreover, we may assume that C' and C'' intersect C in two different points. Indeed, otherwise blow up the common intersection point and blow down the strict transform of C, and repeat this procedure until C' and C'' intersect $\bar{f}^{-1}(\infty)$ in two different points. Thus we get an SNC-completion Y with $Y \setminus S = C \cup C' \cup C'' \cup C_1 \cup \ldots \cup C_l$ such that $C \cdot C = 0$, $C \cdot C' = 1$, $C \cdot C'' = 1$ and C is disjoint from $C_1 \cup \ldots \cup C_l$. Assume again that the dual graph Γ_Y is not linear. Then for a vertex of degree ≥ 3 all but two branches are contractible, see Proposition 4.2(2). But by pseudo-minimality a contractible branch must contain one of the curves C' or C''. However, then it also contains the zero vertex corresponding to C and hence it is not contractible. So we have the following picture (note that D_2 and D_{-2} may or may not be in the same fiber)



and thus $\Gamma_Y = [[\ldots, \eta_{-2}, a, 0, b, \eta_2, \ldots]]$. This completion may be transformed by modifications (B) into the desired form.

Claim (2) follows from the fact that the graph $\tilde{\Gamma}_Y = [[\eta_{-m}, \ldots, \eta_n]]$ can be contracted to a minimal graph $\tilde{\Gamma} = [[\omega_2, \ldots, \omega_k]]$ such that at least one endvertex of $\tilde{\Gamma}_Y$ does not get contracted. Indeed $\tilde{\Gamma}_Y$ has at most one (-1)-vertex. If the right endvertex D_n is not contracted then move the zeros in Γ_Y to the left $[[0, 0, \eta_{-m}, \ldots, \eta_n]]$, and then make the contraction by only inner blow downs onto a completion with dual graph $[[0, \omega_1, \omega_2, \ldots, \omega_k]]$. If the left endvertex D_{-m} is not contracted then repeat the same procedure by moving the zeros to the right $[[\eta_{-m}, \ldots, \eta_n, 0, 0]]$. $\square$

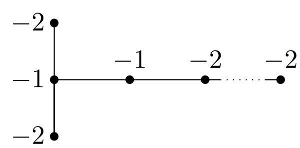
The ω_1-standard completion from the above proposition can be transformed into a standard completion by modifications (A) and there are ω_1 parameters occuring in this process. Therefore we get the following corollary:

Corollary 3.6. *Let S be a Gizatullin surface such that each standard completion is determined by its dual graph*[2]*. Then the family of $\mathbb{C}^*$-fibrations having a pseudo-minimal SNC-completion with a given dual graph that is obtained as in Proposition 3.5 from a ω_1-semistandard completion has at most ω_1 parameters.*

Let us take a closer look how the fibers of a $\mathbb{C}^*$-fibration $f : S \to \mathbb{C}$ with two sections at the boundary can look like. For simplicity assume that the surface S is smooth. Clearly every fiber $f^{-1}(a)$ has precisely one connected component isomorphic to $\mathbb{C}^*$ or to $\mathbb{C} \vee \mathbb{C}$ (two lines intersecting transversally in one point) namly the one connecting $D_{-m} \cup \ldots \cup D_{-1}$ to $D_1 \cup \ldots \cup D_n$. All other connected components are isomorphic to $\mathbb{C}$, clearly all these $\mathbb{C}$ components are adjecent to a curve $D_{-m}, \ldots, D_{-2}, D_2 \ldots D_n$. By Proposition 3.1(b) the total number of $\mathbb{C}$ and $\mathbb{C} \vee \mathbb{C}$ components is equal to $\chi(S)$.

The next proposition will clarify the last possibility, namely when there is a double-section at infinity.

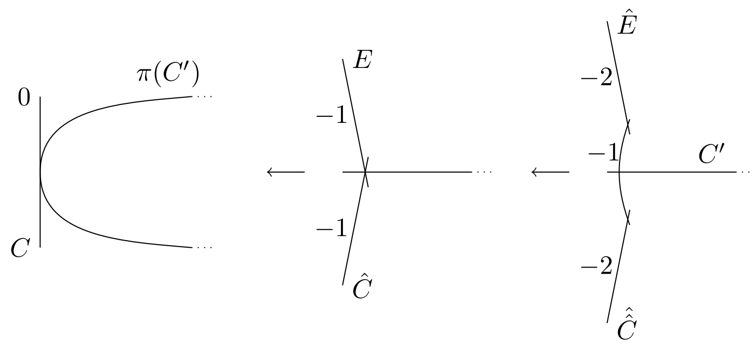
Proposition 3.7. *Let $f : S \to \mathbb{C}$ be a $\mathbb{C}^*$-fibration on a Gizatullin surface S, and let X be a pseudo-minimal SNC-completion of S such that $D = X \setminus S$ contains a double-section C. Then Γ_X is of the form:*



In particular this situation only occurs when the dual graph of a standard completion of S is of the form

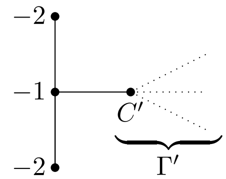
$$[[0, 0, -4]], \quad [[0, 0, -3, -3]] \quad \textit{or} \quad [[0, 0, -3, -2, \ldots, -2, -3]].$$

Proof. Let C' be the double-section. There is a contraction π such that the set $\bar{f}^{-1}(\infty)$ is contractible to a curve C with $C \cdot \pi(C') = 2$ so the curves C and $\pi(C')$ do not intersect transversally, indeed otherwise they would intersect in two points and the dual graph Γ_X would contain a loop. We can see that the dual graph of $\bar{f}^{-1}(\infty)$ is $[[-2, -1, -2]]$ and the double-section C' intersects the (-1)-curve transversally. Indeed after two blow ups the boundary is a SNC-divisor:



So Γ_X is of the form:

[2]A criterion for this property can be extracted from [9]. It applies to Danielewski surfaces.



Since, by Proposition 4.2(2), Γ_X can be transformed into a linear graph the branch Γ' is contractible and by pseudo-minimality the only (-1)-curve in Γ' is C'. This shows that Γ_X is of the desired form. After the contraction of Γ we get a dual graph of the form $[[-2, n, -2]]$ with $n \geq 0$ and they all lead to a standard completion as in the claim. $\square$

Remark 3.8. In [12, Lemmas 4.7+4.8] the $\mathbb{C}^*$-fibrations on affine toric surfaces were classified using other techniques. Affine toric surfaces are Gizatullin surfaces and some of them have a completion as in Proposition 3.7. Therefore it is expected that they have a $\mathbb{C}^*$-fibration that, in some sense, looks essentially different from the other $\mathbb{C}^*$-fibrations. In fact it is possible to see that the twisted $\mathbb{C}^*$-fibrations of affine toric surfaces correspond exactly to the special case appearing in the end of Lemma 4.8. in [12].

We conclude this section by the classification of $\mathbb{C}^*$-fibrations on $\mathbb{C}^2$. This result is well known: Brunella used it for his classification of complete vector fields on $\mathbb{C}^2$ in [6]. He cites Suzuki [15]. Here we give an alternative proof using Lemma 3.5.

Proposition 3.9 ([15])**.** *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a $\mathbb{C}^*$-fibration. Then (up to an automorphism) $f(x, y)$ is of the form $x^i(x^l y - Q(x))^j$ for i, j relatively prime numbers, $l \in \mathbb{N}_0$ and a polynomial Q with* $\deg(Q) < l$*.*

Proof. Since the dual graph of a standard completion is $[[0, 0]]$, and hence not as the ones in Proposition 3.7 there is a pseudo-minimal SNC-completion Y as in Proposition 3.5 with

$$\Gamma_Y = \quad \overset{\eta_{-m}}{\underset{D_{-m}}{\bullet}} \cdots\cdots \overset{\eta_{-2}}{\underset{D_{-2}}{\bullet}} \!-\!\!-\!\!-\! \overset{0}{\underset{D_{-1}}{\bullet}} \!-\!\!-\!\!-\! \overset{0}{\underset{D_{0}}{\bullet}} \!-\!\!-\!\!-\! \overset{\eta_{1}}{\underset{D_{1}}{\bullet}} \!-\!\!-\!\!-\! \overset{\eta_{2}}{\underset{D_{2}}{\bullet}} \cdots\cdots \overset{\eta_{n}}{\underset{D_{n}}{\bullet}} \; .$$

Since $\chi(\mathbb{C}^2) = 1$ there is precisely one $\mathbb{C}$ or one cross of two lines $\mathbb{C} \vee \mathbb{C}$ inside a fiber (say $f^{-1}(0)$) of f. If it is $\mathbb{C} \vee \mathbb{C}$ then by the Abhyankar-Moh-Suzuki theorem we might assume that the zero set of f is $\{x = 0\} \cup \{y = 0\}$. Hence f is of the form $x^i y^j$ ($l = 0$). If it is a $\mathbb{C}$ component (say F_1) then it is attached say to one of the curves $D_2, \ldots, D_n$ and let F_2 be the $\mathbb{C}^*$ component of this fiber. By the absence of other $\mathbb{C}$ components we know that $\bar{F}_2$ intersects D_{-m} since otherwise $D_{-m} \cup \ldots \cup D_{-2}$ would contain a (-1)-curve. By Proposition 3.5(2) we get another completion X of S with

$$\Gamma_X = \quad \overset{0}{\underset{C_0}{\bullet}} \!-\!\!-\!\!-\! \overset{\omega_1}{\underset{C_1}{\bullet}} \; .$$

It is obtained from Y such that $\bar{F}_1$ is disjoint from C_0 and D_{-m} maps isomorphically onto C_0. Thus $\bar{F}_2$ still intersects C_0 transversally in one point. We continue by blowing up the point $C_0 \cap \bar{F}_2$ and blowing down the strict transform of C_0, which is a modification of type (A). Repeating this ω_1-times we will end up with a completion isomorhic to $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\overline{F}_2$ intersects $\{x = \infty\}$ transversally in one point, and $\overline{F}_1 \cap \{x = \infty\} = \emptyset$. Hence in these coordinates we may assume that $F_1 = \{x = 0\}$ and $F_2 = \{y = R(x)\}$ is a graph for a rational function R with a pole at 0. Let us write R as $R(x) = Q(x)x^{-l} + P(x)$ for some polynomials Q and P. Then after a coordinate change given by $(x, y) \mapsto (x, y + P(x))$ we get that $F_2 = \{x^l y = Q(x)\}$, and thus the claim follows. $\square$

4. $\mathbb{C}$- and $\mathbb{C}^*$-fibrations on smooth Danielewski surfaces

Danielewski surfaces form a subfamily of Gizatullin surfaces. They have an explicit description as a hypersurface in $\mathbb{C}^3$ and the classification of $\mathbb{C}$- and $\mathbb{C}^*$- fibrations can be done very explicit. In most cases the classification looks exactly the same as the classification of $\mathbb{C}$- and $\mathbb{C}^*$- fibrations on $\mathbb{C}^2$. It is a direct consequence of the famous Abhyankar-Moh-Suzuki theorem that all $\mathbb{C}$-fibrations are up to an automorphism given by the projection to the x-coordinate. Actually, this classification has already been found by Gutwirth [10]. Proposition 3.9 is the description of $\mathbb{C}^*$-polynomials on $\mathbb{C}^2$.

Definition 4.1. A smooth affine surface S in called *Danielewski surface*[3] if there is an SNC-completion X such that $\Gamma_X = [[0, 0, -k]]$ for $k \geq 2$. Danielewski surfaces can also be seen as surfaces in $\mathbb{C}^3$ given by the equation $\{xy = p(z)\}$ for a polynomial p of degree k with simple zeros.

Let p be a polynomial of degree k with simple zeros. Given the surface $S = \{xy = p(z)\} \subset \mathbb{C}^3$ it is easy to construct a standard completion. The projection $\pi(x, y, z) = (x, z)$ is a birational map from S to $\mathbb{C}^2$, it is an isomorphism on the open sets $\{x \neq 0\}$ and it contracts the lines $\{x = 0, z = z_i\}$ onto the points $(0, z_i)$ where the numbers z_i are the zeros of p. So S is isomorphic to an open set in $\mathbb{C}^2$ blown up in the points $(0, z_i)$ and therefore $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in these points is a completion X_0 of S. The curve

$$D_0 = X_0 \setminus S = \{\widehat{x = \infty}\} \cup \{\widehat{z = \infty}\} \cup \{\widehat{x = 0}\}$$

is the boundary with dual graph $\Gamma_{X_0} = [[0, 0, -k]]$ (where $\hat{C}$ denotes the strict transform of a curve C). Moreover, the projection to the x (resp. z) coordinate corresponds to the map ϕ_0 (resp. ϕ_1) constructed in the previous section and therefore the map π corresponds to the map ϕ.

On the other hand, given any standard completion of a Danielewski surface S, its corresponding map ϕ will describe a way to embed S into $\mathbb{C}^3$. Indeed, S is given in $\mathbb{C}^3$ by the equation $xy = p(z)$, when the polynomial p is defined such that its zeros are the indeterminacy points of ϕ^{-1}).

We begin with the description of $\mathbb{C}$-fibrations on S:

Proposition 4.2 ([3, 7, 9, 13])**.** *Let $f : S = \{xy = p(z)\} \to \mathbb{C}$ be a $\mathbb{C}$-fibration.*

(1) Up to automorphism of S the fibration f is given by the projection $f(x, y, z) = x$.

(2) Any standard completion of S is isomorphic to the standard completion X_0 constructed above.

[3]In the literature surfaces given by $\{x^n y = p(z)\}$ are often also called Danielewski surfaces. In this text we only consider the case $n = 1$.

Proof. By Corollary 3.4 (1) is equivalent to (2). There are several proofs, e.g. (1) is proven in [13] and (2) is proven in [9]. □

4.1. **$\mathbb{C}^*$-fibrations with two sections at the boundary.** The description of $\mathbb{C}^*$-fibration with two sections at the boundary is very much related with the description of $\mathbb{C}^*$-fibrations on $\mathbb{C}^2$. We will prove the following proposition:

Proposition 4.3. *Let $f : S = \{xy = p(z)\} \to \mathbb{C}$ be a $\mathbb{C}^*$-fibration with two sections at the boundary. Then f is up to isomorphism of S of the form z or $x^i(x^l(z+a)+Q(x))^j$ for i,j relatively prime, $l \in \mathbb{N}_0$, $\deg(Q) < l$ and $a \in \mathbb{C}$.*

Proof. Let $X \supset S$ be the semistandard completion from Proposition 3.5 with

$$\Gamma_X = \quad \overset{0}{\underset{C_0}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{\omega_1}{\underset{C_1}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{-k}{\underset{C_2}{\bullet}}$$

that is obtained (starting by moving the zeros to the left followed by inner blow downs) from a pseudo-minimal SNC-completion Y with

$$\Gamma_Y = \quad \overset{\eta_{-m}}{\underset{D_{-m}}{\bullet}} \cdots\cdots \overset{\eta_{-2}}{\underset{D_{-2}}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{0}{\underset{D_{-1}}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{0}{\underset{D_0}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{\eta_1}{\underset{D_1}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{\eta_2}{\underset{D_2}{\bullet}} \cdots\cdots \overset{\eta_n}{\underset{D_n}{\bullet}}$$

with $\eta_i \leq -2$ for $|i| \geq 2$ and $\eta_{-1} \leq -1$. Since $[[\eta_{-m}, \ldots, \eta_{-2}, \eta_1, \ldots, \eta_n]]$ is contractible to $[[-k]]$ such that the right endvertex is not contracted we have $\eta_1 = -1$ and thus $n \geq 2$ unless $X = Y$. We may extend $f : S \to \mathbb{C}$ to a rational function $\bar{f} : X \dashrightarrow \mathbb{P}^1$. If $X = Y$ then $f(x,y,z) = z$ up to isomorphism. Indeed, by Proposition 4.2 the completion $X = Y$ is isomorphic to X_0, and $\bar{f} : X \cong X_0 \to \mathbb{P}^1$ coincides with ϕ_1 which is the projection to the z-coordinate. If $X \neq Y$ then by construction $\bar{f}$ is constant and non-polar on $C_2 \setminus C_1$ (assume that f vanishes on $C_2 \setminus C_1$). Indeed, C_2 is the strict transform of D_n which sits inside a fiber (since $n \geq 2$). The same holds true if we pass by modifications (A) to a standard completion X'. Hence the pushforward $\phi_* \bar{f}$ by the morphism $\phi : X' \cong X_0 \to \mathbb{P}^1 \times \mathbb{P}^1$ restricted to $\mathbb{C}^2$ is a regular function $g := \phi_* \bar{f}|_{\mathbb{C}^2} : \mathbb{C}^2 \to \mathbb{C}$. In particular, g is a polynomial function on $\mathbb{C}^2$ with general fibers isomorphic to $\mathbb{C}^*$ and $\{x = 0\} \subset g^{-1}(0)$. By Proposition 3.9 the function g and hence its pullback f is, for some automorphism (s,t) of $\mathbb{C}^2$, of the form $s(x,z)^i(s(x,z)^l t(x,z) - Q(s(x,z)))^j$ with i,j,l,Q as desired. Clearly we have that (if $l = 0$ then maybe after exchanging s and t) the zero set of $s(x,z)$ coincides with $\{x = 0\}$. Hence the automorphism is of the form $s(x,z) = ax$ and $t(x,z) = by + r(x)$, and after rescaling f we may assume $a = b = 1$. Since automorphisms of $\mathbb{C}^2$ of the form $(x,z) \mapsto (x, z + xr'(x))$ extend to the surface S we may even assume that $s(x,z) = x$ and $t(x,z) = z + a$ for some $a \in \mathbb{C}$ and the claim follows. □

4.2. **$\mathbb{C}^*$-fibrations with one double-section at the boundary.** By Proposition 3.7 the case of a $\mathbb{C}^*$-fibration with a double-section at the boundary on a Danielewski surface only occurs when the polynomial p is of degree 4. It will be more convenient to allow completions where the components of the boundary do not necessarily intersect transversally.

Lemma 4.4. *Let X be a non-SNC-completion of $S = \{xy = a(z-z_1)(z-z_2)(z-z_3)(z-z_4)\}$ such that $X \setminus S = C_0 \cup C_1$ with $C_0 \cdot C_1 = 2$, $C_0 \cdot C_0 = 0$ and $C_1 \cdot C_1 = 1$. Then:*

(1) X can be identified with $\mathbb{P}^2$ blown up in $[z_i^2 : z_i : 1]$ for $1 \leq i \leq 4$ such that

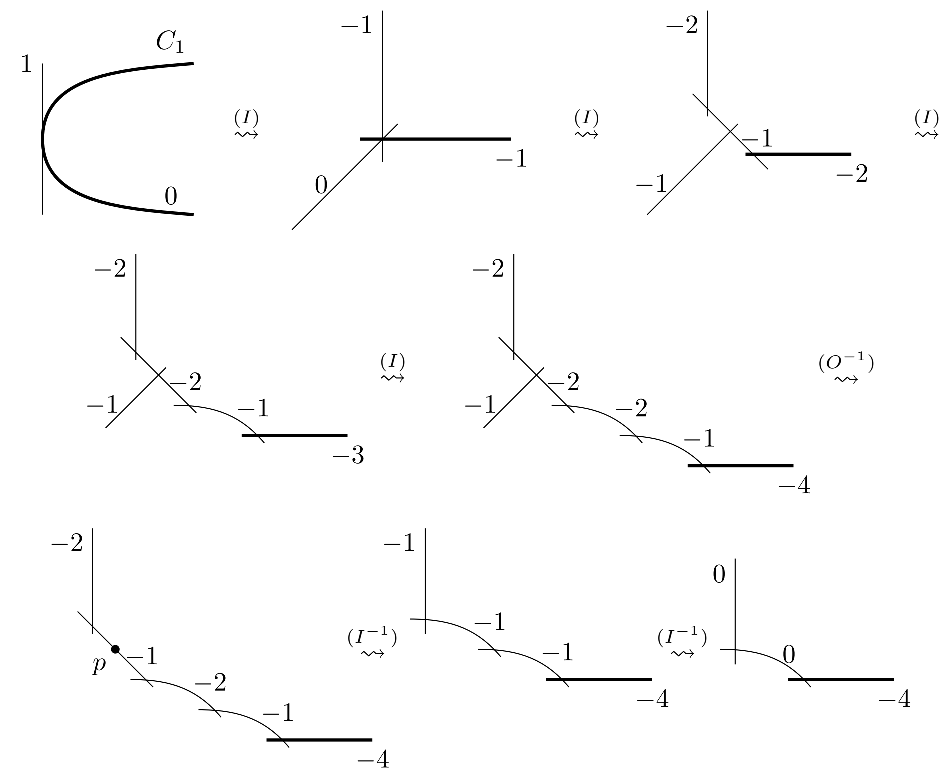
$$C_0 = \{\widehat{w = 0}\} \quad \text{and} \quad C_1 = \{\widehat{uw = v^2}\}.$$

(2) There is a unique (up to affine transformation of $\mathbb{P}^1 \setminus \{\infty\}$) rational function $h : X \to \mathbb{P}^1$ such that C_0 is a double-section and $C_1 = h^{-1}(\infty)$. The pushforward $\tilde{h}$ of h to $\mathbb{P}^2$ is given by

$$\tilde{h}([u : v : w]) = \frac{(u - (z_1 + z_2)v + z_1 z_2 w)(u - (z_3 + z_4)v + z_3 z_4 w)}{uw - v^2}.$$

Moreover, h has at least three fibers which are not isomorphic to $\mathbb{C}^*$.

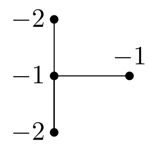
Proof. The completion X may be transformed into a standard completion by the following modifications:



A calculation shows that the birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ given by $(x, z) \mapsto [u(x, z) : v(x, z) : w(x, z)] = [x + az^2 : z : 1]$ induces precisely the inverse of this modification on the boundary (where a corresponds to the point p). Thus X can be identified with $\mathbb{P}^2$ blown up in $[az_i^2 : z_i : 1]$ for $1 \leq i \leq 4$ (indeed, the standard completion was isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in $(0, z_i)$ for $1 \leq i \leq 4$). After the isomorphism $[u : v : w] \mapsto [a^{-1}u : v : w]$ the completion X is as desired. For Claim (2) we observe that $\tilde{h}$ is of degree 2, indeed the general fiber meets $\{w = 0\}$ twice. Moreover, every fiber meets $\{uw = v^2\}$ precisely in the points $[z_i^2 : z_i : 1]$ for $1 \leq i \leq 4$. This holds since these points are indeterminacy points of $\tilde{h}$ because C_1 is an entire fiber of h. The space of curves of degree 2 in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^5$ hence the space of curves of degree 2 passing through four points is isomorphic to $\mathbb{P}^1$. It coincides with the levels of $\tilde{h}$. So, $\tilde{h}$ is (up to affine transformation of $\mathbb{P}^1 \setminus \{\infty\}$) of the form $(uw - v^2)^{-1} g(u, v, w)$, where g is any homogeneous polynomial of degree 2 such that its zero set meets $\{uw = v^2\}$ in the four requested points. We may choose the product of two linear functions each connecting two of the points linearly. Clearly h has at least three fibers not isomorphic to $\mathbb{C}^*$ since there are three possibilities to choose two lines through these four points. $\square$

Proposition 4.5. *Let $f : S = \{xy = p(z)\} \to \mathbb{C}$ be a $\mathbb{C}^*$-fibration with double-section at the boundary. Then $\deg p = 4$ and f is given up to automorphism of S by $f(x, y, z) = ax + y + \frac{1}{6}p''(z)$ where a is the leading coefficient of p. Additionally, the fibration f has at least three fibers not isomorphic to $\mathbb{C}^*$.*

Proof. By Proposition 3.7 the polynomial p has degree 4 (say $p(z) = a(z - z_1)(z - z_2)(z - z_3)(z - z_4)$), and moreover there is a pseudo-minimal SNC-completion with dual graph of the boundary



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This completion can be transformed by two blow downs into a completion X as in Lemma 4.4, which is then by (1) isomorphic to $\mathbb{P}^2$ blown up in 4 points. The birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ given by $(x, z) \mapsto [u(x, z) : v(x, z) : w(x, z)] = [x + z^2 : z : 1]$ induces a birational map from the standard completion X_0 to the completion X. By (2) of Lemma 4.4 the fibration f is given by

$$\begin{aligned}
\frac{(u - (z_1 + z_2)v + z_1 z_2 w)(u - (z_3 + z_4)v + z_3 z_4 w)}{uw - v^2} &= \\
\frac{(x + z^2 - (z_1 + z_2)z + z_1 z_2)(x + z^2 - (z_3 + z_4)z + z_3 z_4)}{x + z^2 - z^2} &= \\
\frac{1}{x}\left[\begin{array}{c} x^2 + x(2z^2 - (z_1 + z_2 + z_3 + z_4)z + z_1 z_2 + z_3 z_4) \\ +(z - z_1)(z - z_2)(z - z_3)(z - z_4) \end{array}\right] &= \\
x + \frac{p(z)}{ax} + 2z^2 - (z_1 + z_2 + z_3 + z_4)z + z_1 z_2 + z_3 z_4 &= \\
x + \frac{y}{a} + 2z^2 - (z_1 + z_2 + z_3 + z_4)z + z_1 z_2 + z_3 z_4 &
\end{aligned}$$

and hence f is (after multiplying with a and adding a constant) of the desired form. $\square$

5. Proof of the Main Theorem

Let p be a polynomial with simple zeros, and let

$$\nu = \nu_x(x, y, z)\frac{\partial}{\partial x} + \nu_y(x, y, z)\frac{\partial}{\partial y} + \nu_z(x, y, z)\frac{\partial}{\partial z}$$

be a complete algebraic vector field on the Danielewski surface $S = \{xy = p(z)\}$ extended regularly to $\mathbb{C}^3$. Then, as mentioned in the Introduction, by [11, Theorem 1.3] the vector field ν preserves a $\mathbb{C}$- or $\mathbb{C}^*$-fibration $f : S \to \mathbb{C}$. These fibrations are described in the previous section. Hence it is possible to give the precise form of ν using exactly the same arguments as in the planar case (see Proposition 2 in [6]). Let us establish first two lemmas.

Lemma 5.1. *Assume that ν is tangent to $\{x = 0\}$. Then ν projects to a complete vector field*

$$\nu_x(x, \frac{p(z)}{x}, z)\frac{\partial}{\partial x} + \nu_z(x, \frac{p(z)}{x}, z)\frac{\partial}{\partial z}$$

on $\mathbb{C}^*_x \times \mathbb{C}_z$, ν_x and ν_z are divisible by x and ν is of the form

$$\nu = \frac{\nu_x(x, y, z)}{x}\mathrm{HF} + \frac{\nu_z(x, y, z)}{x}\mathrm{SF}^x.$$

Proof. Regarding ν as a derivation, clearly $\nu_x = \nu(x)$ vanishes on $\{x = 0\}$. Therefore we have $\nu(z)p'(z) = \nu(p(z)) = \nu(xy) = x\nu(y) + y\nu(x) = 0$ for $x = 0$. Hence $\nu_z = \nu(z)$ vanishes also on $\{x = 0\}$. This means that both ν_x and ν_z are divisible by x. Thus we get the explicit form of $\nu_y = \nu(y) = (p'(z)\nu_z - y\nu_x)/x$, and also the explicit form of ν as in the claim. □

Lemma 5.2 ([6])**.** *(1) Let $D_\alpha \times C_t$ be a (holomophic) trivialization of a neighborhood of a general fiber C of f. Then the pullback of ν to this neighborhood is of the form*

$$\tilde{\nu} = F(\alpha)\frac{\partial}{\partial \alpha} + (G(\alpha)t + H(\alpha))\frac{\partial}{\partial t}$$

for holomorphic functions F, G and H. If $C \cong \mathbb{C}^*$ then $H = 0$.

(2) If $\nu = \nu_1 + \nu_2$, where ν_2 is complete and tangent to the fibers of f. Then ν_1 is complete.

Proof. Since the local flow of $\tilde{\nu}$ sends vertical fibers to vertical fibers the first summand of $\tilde{\nu}$ is independent of t. By the Riemann removable singularities theorem the local flow maps of $\tilde{\nu}$ extends to maps $\{\alpha\} \times \bar{C} \to \{\alpha'\} \times \bar{C}$. Hence $\tilde{\nu}$ extends to $D \times \bar{C}$ such that $\tilde{\nu}$ is tangential to $D \times \partial C$. Thus the second summand is of the desired form. The second claim follows from the fact that also ν_1 extends to $\bar{C}$ such that it is tangential to the sections at infinity. □

These two lemmas directly imply the next proposition concerning the case of $\mathbb{C}$-fibrations.

Proposition 5.3. *If $f(x, y, z) = x$ then*

$$\nu = c\mathrm{HF} + (A(x)z + B(x))\,\mathrm{SF}^x$$

for some $c \in \mathbb{C}$ and $A, B \in \mathbb{C}[x]$.

Proof. Since $\{x = 0\}$ is a singular fiber, ν is tangential to it. Lemma 5.1 shows that it is sufficient to look at the projection and restriction of ν to $\mathbb{C}^*_x \times \mathbb{C}_z$. The latter is obviously a trivialization of a neighborhood of a fiber. Hence Lemma 5.2(1) shows that ν is of the form $F(x)\partial/\partial x + (G(x)z + H(x))\,\partial/\partial z$ on $\mathbb{C}^*_x \times \mathbb{C}_z$. By Lemma 5.1 the functions F, G, H are divisible by x. By the completeness of ν we have $F(x) = cx$ for some c, which leads to the desired form. □

The $\mathbb{C}^*$-case with two sections at the boundary works similarly. The only new difficulty is to trivialize a neighborhood of a fiber.

Proposition 5.4. *If $f(x,y,z) = x^m(x^l(z+a)+Q(x))^n$ for coprime numbers $m,n \in \mathbb{N}$, $l \in \mathbb{N}_0$, $a \in \mathbb{C}$ and $\deg(Q) < l$ then*

$$\begin{aligned}\nu &= c\left(\frac{z+a}{x} + \frac{Q(x)}{x^{l+1}}\right)\mathrm{SF}^x + A(x^m(x^l(z+a)+Q(x))^n) \\ &\cdot \left[n\mathrm{HF} - \left(\frac{(m+nl)(z+a)}{x} + \frac{mQ(x)+nxQ'(x)}{x^{l+1}}\right)\mathrm{SF}^x\right]\end{aligned}$$

for some $c \in \mathbb{C}$ and $A \in \mathbb{C}[t]$ satisfying $A(0) = c/(m+nl)$ and $A(x^m(x^l(z+a)+Q(x))^n)(mQ(x)+nxQ'(x)) - cQ(x) \in x^{l+1}\cdot\mathbb{C}[S]$.

Proof. Again ν is tangential to $\{x=0\}$, so we work on $\mathbb{C}_x^* \times \mathbb{C}_z$ as in Lemma 5.1. Pick $0 \neq \alpha_0 \in \mathbb{C}$, and let $D = \{|\alpha - \alpha_0| < \varepsilon\}$ be a small ball around α_0. Then the map

$$\begin{aligned} D \times \mathbb{C}^* &\to \mathbb{C}_x^* \times \mathbb{C}_z \\ (\alpha, t) &\mapsto \left(t^n, \frac{e^\alpha t^{-m} - Q(t^n)}{t^{nl}} - a\right)\end{aligned}$$

gives a trivialization of a neighborhood of the fiber $f^{-1}(e^{n\alpha_0})$. Using this map yields:

$$\begin{aligned}\frac{\partial}{\partial\alpha} &\mapsto \nu_1 := \left(z + a + \frac{Q(x)}{x^l}\right)\frac{\partial}{\partial z}, \\ t\frac{\partial}{\partial t} &\mapsto \nu_2 := nx\frac{\partial}{\partial x} - \left((m+nl)(z+a) + \frac{mQ(x)+nxQ'(x)}{x^l}\right)\frac{\partial}{\partial z}.\end{aligned}$$

Lemma 5.2(1) shows that ν is given on $\mathbb{C}_x^* \times \mathbb{C}_z$ by $F(\alpha)\nu_1 + G(\alpha)\nu_2$ for $\alpha = x^m(x^l(z+a)+Q(x))^n$. We know that $G(\alpha)\nu_2$ is complete on $\mathbb{C}_x^* \times \mathbb{C}_z$ since it is tangent along the fibers of f and its restriction to any fiber is complete. Thus by Lemma 5.2(2) also $F(\alpha)\nu_1$ is complete on $\mathbb{C}_x^* \times \mathbb{C}_z$. This shows that $F(\alpha)$ is constant. Letting $A = G$ yields that ν is as desired on $\mathbb{C}_x^* \times \mathbb{C}_z$. Lemma 5.1 provides a lift of ν to the vector field on S as in the claim. In order to be non-polar on $\{x=0\}$ we need the additional condition on A, which is equivalent to the fact that ν_z is divisible by x. □

Proposition 5.5. *If $p(z) = a\cdot(z^4+bz^3+cz^2+dz+e)$ and $f(x,y,z) = ax+y+\frac{1}{6}p''(z)$ then*

$$\nu = A\left(ax+y+\frac{1}{6}p''(z)\right)\left(-\frac{1}{6}p'''(z)\mathrm{HF} + a\mathrm{SF}^x - \mathrm{SF}^y\right)$$

for some $A \in \mathbb{C}[t]$.

Proof. By Proposition 4.5 we know that f has more than one fiber not isomorphic to $\mathbb{C}^*$. Thus ν acts on the base $\mathbb{C}$ with more that one fixed point. By hyperbolicity ν is tangential to the fibers of f. Hence ν restricted to a general fiber is proportional to $t\partial/\partial t$. We need to parametrize a general fiber $C^\alpha = \{ax+y+2az^2+abz+a\alpha = 0\}$, $\alpha \in \mathbb{C}$. Let us define $\xi, \chi, \kappa \in \mathbb{C}$ such that

$$\xi^2 = \alpha + \frac{b^2}{2} - c, \quad \chi = \frac{\alpha b - 2d}{4\xi^2}, \quad \kappa = e - \frac{\alpha^2}{4} + \xi^2\chi^2.$$

The map $C^\alpha \to \mathbb{C}^*$ defined by

$$(x,y,z) \mapsto t := x + z^2 + \frac{b}{2}z + \frac{\alpha}{2} + \xi(z+\chi) = \frac{ax-y}{2a} + \xi(z+\chi)$$

is an isomorphism. Indeed, after multiplying with x/a and replacing xy by $p(z)$ the equation defining C^α becomes

$$\begin{aligned} x^2+z^4+bz^3+cz^2+dz+e+(2z^2+bz+\alpha)x &= \\ \left(x+z^2+\frac{b}{2}z+\frac{\alpha}{2}\right)^2+\left(c-\frac{b^2}{4}-\alpha\right)z^2+\left(d-\frac{\alpha b}{2}\right)z+e-\frac{\alpha^2}{4} &= \\ \left(x+z^2+\frac{b}{2}z+\frac{\alpha}{2}\right)^2-(\xi(z+\chi))^2+\kappa &= \\ t(t-2\xi(z+\chi))+\kappa &. \end{aligned}$$

Thus t can be seen as a variable of $\mathbb{C}^*$. Moreover, we can see that the vector field $\nu_0=-\frac{1}{6}p'''(z)\mathrm{HF}+a\mathrm{SF}^x-\mathrm{SF}^y$ is tangent to the fibers and restricts to the vector field $2a\xi t\partial/\partial t$ on $C^\alpha\cong\mathbb{C}^*$. Indeed, ν_0 acts on t by multiplication with $2a\xi$:

$$\begin{aligned} \nu_0(t) &= \nu_0\left(\frac{ax-y}{2a}+\xi(z+\chi)\right) \\ &= 2a\xi\left(-\frac{p'''(z)}{6}\cdot\frac{ax+y}{4a^2\xi}-\frac{p'(z)}{2a\xi}+\frac{ax-y}{2a}\right) = \\ 2a\xi&\left(-\frac{1}{\xi}\left((4z+b)\frac{-(2z^2+bz+\alpha)}{4}+2z^3+\frac{3}{2}bz^2+cz+\frac{d}{2}\right)+\frac{ax-y}{2a}\right) \\ &= 2a\xi\left(-\frac{1}{\xi}\left(\left(-\frac{b^2}{4}-\alpha+c\right)z-\frac{\alpha b}{4}+\frac{d}{2}\right)+\frac{ax-y}{2a}\right) \\ &= 2a\xi\left(\xi(z+\chi)+\frac{ax-y}{2a}\right) \\ &= 2a\xi t \end{aligned}$$

Overall on every fiber of f the vector field ν is a multiple of ν_0. Thus the proposition is proven. □

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